Lecture Notes, January 11, 2012

Mathematical Logic

Logical Inference

Let A and B be two logical conditions, like A="it's sunny today" and B="the light outside is very bright"

A $\Rightarrow$ B

A implies B, if A then B

A $\Leftrightarrow$ B

A if and only if B, A implies B and B implies A, A and B are equivalent conditions

Proofs

Just like in high school geometry.

Concept of Proof by contradiction: Suppose we want to show that **A** $\Rightarrow$ B. Ordinarily, we'd like to prove this directly. But it may be easier to show that [not (**A** $\Rightarrow$ B)] is false. How? Show that [A & (not B)] leads to a contradiction. A: x = 1, B:x+3=4. Then [A & (not B)] leads to the conclusion that $1+3\neq 4$ or equivalently $1\neq 1$, a contradiction. Hence [A & (not B)] must fail so A$\Rightarrow$ B. (Yes, it does feel backwards, like your pocket is being picked, but it works).

Set Theory

Definition of a Set

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{x | x has property P}

{1, 2, ..., 9, 10} = { x | x is an integer, $1\leq x \leq 10$ }.

Elements of a set

$x \in A$; $y \notin A$

$x \neq \{ x \}$

$x \in \{ x \}$

$\phi \equiv$ the empty set ($\equiv$ null set), the set with no elements.

Subsets

$A \subset B$ or $A \subseteq B$ if $x \in A \Rightarrow x \in B$

$A \subset A$ and $\phi \subset A$.

Set Equality

A = B if A and B have precisely the same elements

A = B if and only if $A \subset B$ and $B \subset A$.

Set Union

$A \cup B$

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ ('or' includes 'and')

Set Intersection

$\cap$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

If $A \cap B = \phi$ we say that A and B are disjoint.

Theorem 6.1: Let A, B, C be sets,

a.	$A \cap A = A,\ A \cup A = A$	(idempotency)
b.	$A \cap B = B \cap A,\ A \cup B = B \cup A$	(commutativity)
c.	$A \cap (B \cap C) = (A \cap B) \cap C$	(associativity)
	$A \cup (B \cup C) = (A \cup B) \cup C$	
d.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(distributivity)
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	

Complementation (set subtraction)

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$A \backslash B = \{x \mid x \in A,\ x \notin B\}$

Cartesian Product

<u>ordered pairs</u>

$A \text{ x } B = \{(x, y) \mid x \in A,\ y \in B\}$.

Note: If $x \neq y$, then $(x, y) \neq (y, x)$.

R = The set of real numbers

$\mathbf{R}^N$ = N-fold Cartesian product of R with itself.

$\mathbf{R}^N$ = R x R x R x ... x R, where the product is taken N times.

The order of elements in the ordered N-tuple (x, y, ...) is essential. If $x \neq y,\ (x, y, \ldots) \neq (y, x, \ldots)$.

$\mathbf{R}^N$, Real N-dimensional Euclidean space

Read Starr's *General Equilibrium Theory*, Chapter 7.

R^2 = plane

R^3 = 3-dimensional space

R^N = N-dimensional Euclidean space

Definition of R:

R = the real line

$\pm\infty \notin R$

+, - , × , ÷

closed interval : $[a, b] \equiv \{x | x \in R, a \le x \le b\}$.

R is *complete*. Nested intervals property: Let $x^\nu < y^\nu$ and $[x^{\nu+1}, y^{\nu+1}] \subseteq [x^\nu, y^\nu]$, $\nu = 1, 2, 3, \ldots$. Then there is $z \in R$ so that $z \in [x^\nu, y^\nu]$, for all ν .

R^N = N-fold Cartesian product of R.

$x \in R^N$, $x = (x_1, x_2, \ldots, x_N)$

x_i is the ith co-ordinate of x.

x = point (or *vector*) in R^N

Algebra of elements of R^N

$x + y = (x_1 + y_1,\ x_2 + y_2,\ \ldots,\ x_N + y_N)$

0 = (0, 0, 0, ..., 0) , the origin in N-space

$x - y \equiv x + (-y) = (x_1 - y_1, x_2 - y_2, \ldots, x_N - y_N)$

$t \in R,\ x \in R^N$, then $tx \equiv (tx_1,\ tx_2,\ \ldots,\ tx_N)$.

$x,\ y \in R^N,\quad x \cdot y = \sum_{i=1}^{N} x_i y_i$. If $p \in R^N$ is a price vector and $y \in R^N$ is an economic action, then $p \cdot y = \sum_{n=1}^{N} p_n y_n$ is the value of the action y at prices p.

Norm in R^N, the measure of distance

$$|x| \equiv \|x\| \equiv \sqrt{x \cdot x} \equiv \sqrt{\sum_{i=1}^{N} x_i^2} \quad .$$

Let $x,\ y \in R^N$. The distance between x and y is $\|x - y\|$.

$|x - y| = \sqrt{\Sigma_i (x_i - y_i)^2}$.

$\|x - y\| \ge 0$ all $x,\ y \in R^N$

$|x - y| = 0$ if and only if $x = y$.

Limits of Sequences

x^ν , $\nu = 1, 2, 3, \ldots$,

Example: $x^\nu = 1/\nu$. 1, 1/2, 1/3, 1/4, 1/5, ... $x^\nu \to 0$.

Formally, let $x^i \in R,\ i = 1,\ 2,\ \ldots$. Definition: We say $x^i \to x^0$ if for any $\varepsilon > 0$, there is $q(\varepsilon)$ so that for all $q' > q(\varepsilon)$, $|x^{q'} - x^0| < \varepsilon$.

So in the example $x^\nu = 1/\nu$, $q(\varepsilon) = 1/\varepsilon$

Let $x^i \in R^N,\ i = 1,\ 2,\ \ldots$. We say that $x^i \to x^0$ if for each co-ordinate $n = 1,\ 2,\ \ldots,\ N,\ x_n^i \to x_n^0$.

Theorem 7.1: Let $x^i \in R^N,\ i = 1,\ 2,\ \ldots$. Then $x^i \to x^0$ if and only if for any ε there is $q(\varepsilon)$ such that for all $q' > q(\varepsilon)$, $\|x^{q'} - x^0\| < \varepsilon$.

x^o is a *cluster point* of $S \subseteq \mathbf{R}^N$ if there is a sequence $x^\nu \in R^N$ so that $x^\nu \to x^o$.

Open Sets

Let $X \subset R^N$; X is *open* if for every $x \in X$ there is an $\varepsilon > 0$ so that $\|x - y\| < \varepsilon$ implies $y \in X$.

Open interval in R: $(a, b) = \{ x \mid x \in R, a < x < b\}$

ϕ and R^N are open.

Closed Sets

Example: Problem - Choose a point x in the closed interval [a, b] (where $0 < a < b$) to maximize x^2. Solution: $x = b$.

Problem - Choose a point x in the open interval (a, b) to maximize x^2. There is no solution in (a, b) since $b \notin (a, b)$.

A set is closed if it contains all of its cluster points.

Definition: Let $X \subset R^N$. X is said to be a **<u>closed</u>** set if for every sequence x^ν, $\nu = 1, 2, 3, \ldots$, satisfying,

(i) $x^\nu \in X$, and

(ii) $x^\nu \to x^0$,

it follows that $x^0 \in X$.

Examples: A closed interval in R, [a, b] is closed

A closed ball in R^N of radius r, centered at $c \in R^N$, $\{x \in R^N \mid |x-c| \leq r\}$ is a closed set.

A line in R^N is a closed set

<u>But</u> a set may be neither open nor closed (for example the sequence $\{1/\nu\}$, $\nu=1, 2, 3, 4, \ldots$ is not closed in R, since 0 is a limit point of the sequence but is not an element of the sequence; it is not open since it consists of isolated points).

Note: Closed and open are not antonyms among sets. ϕ and R^N are each both closed and open. For a YouTube reference: www.youtube.com/watch?v=SyD4p8_y8Kw

Let $X \subseteq R^N$. The closure of X is defined as

$\bar{X} \equiv \{ y \mid \text{there is } x^\nu \in X, \nu = 1, 2, 3, \ldots , \text{so that } x^\nu \to y \}$.

For example the closure of the sequence in R, $\{1/\nu \mid \nu=1, 2, 3, 4, \ldots \}$ is

$\{0\}\cup\{1/\nu \mid \nu=1, 2, 3, 4, \ldots \}$.

Theorem 7.2: Let $X \subset R^N$. X is closed if $R^N \setminus X$ is open.
Proof: Suppose $R^N \setminus X$ is open. We must show that X is closed. If $X=R^N$ the result is trivially satisfied. For $X \neq R^N$, let $x^\nu \in X$, $x^\nu \to x^o$. We must show that $x^o \in X$ if $R^N \setminus X$ is open. Proof by contradiction. Suppose not. Then $x^o \in R^N \setminus X$. But $R^N \setminus X$ is open. Thus there is an ε neighborhood about x^o entirely contained in $R^N \setminus X$. But then for ν large, $x^\nu \in R^N \setminus X$, a contradiction. Therefore $x^o \in X$ and X is closed. QED

Theorem 7.3: 1. $X \subset \bar{X}$
2. $X = \bar{X}$ if and only if X is closed.

Bounded Sets

Def: $K(k) = \{x \mid x \in R^N,\ |x_i| \le k,\ i = 1,\ 2,\ \ldots,\ N\}$ = cube of side 2k (centered at the origin).

Def: $X \subset R^N$. X is *bounded* if there is $k \in R$ so that $X \subset K(k)$.

Compact Sets

THE IDEA OF COMPACTNESS IS ESSENTIAL!

Def: $X \subset R^N$. X is *compact* if X is closed and bounded.

Finite subcover property: An open covering of X is a collection of open sets so that X is contained in the union of the collection. It is a property of compact X that for every open covering there is a finite subset of the open covering whose union also contains X. That is, every open covering of a compact set has a finite subcover.

Boundary, Interior, etc.

$X \subset R^N$, Interior of X = $\{y \mid y \in X$, there is $\varepsilon > 0$ so that $\|x - y\| < \varepsilon$ implies $x \in X\}$

Boundary $X \equiv \bar{X} \setminus$Interior X

Set Summation in R^N

Let $A \subseteq R^N$, $B \subseteq R^N$. Then

$A + B \equiv \{ x \mid x = a + b, a \in A, b \in B \}$.

The Bolzano-Weierstrass Theorem, Completeness of R^N .

Theorem 7.4 (Nested Intervals Theorem): By an interval in R^N , we mean a set I of the form $I = \{(x_1,\ x_2,\ \ldots,\ x_N) \mid a_1 \le x_1 \le b_1\ ,\ a_2 \le x_2 \le b_2,\ \ldots,\ a_N \le x_N \le b_N,\ a_i,\ b_i \in R\}$.
Consider a sequence of nonempty closed intervals I_k such that

$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_k \supseteq \ldots$.

Then there is a point in R^N contained in all the intervals. That is, $\exists x^o \in \bigcap_{i=1}^{\infty} I_i$ and therefore $\bigcap_{i=1}^{\infty} I_i \neq \phi$; the intersection is nonempty.

Proof: Follows from the completeness of the reals, the nested intervals property on R.

Corollary (Bolzano-Weierstrass theorem for sequences): Let x^i , i = 1, 2, 3, ... be a bounded sequence in R^N . Then x^i contains a convergent subsequence.

Proof 2 cases: x^i assumes a finite number of values, x^i assumes an infinite number of values.

It follows from the Bolzano-Weierstrass Theorem for sequences and the definition of compactness that an infinite sequence on a compact set has a convergent subsequence whose limit is in the compact set.